Symmetries in Group Theory

- One of the most important and beautiful themes unifying many areas of modern mathematics is the study of symmetry.
- Group theory is the mathematical study of symmetry.
- A group can be treated as symmetry of an object.
- The object can be a plane figure, usually a regular polygon like an equilateral triangle, a square, a solid like a regular tetrahedron, etc.

What is a Symmetry?

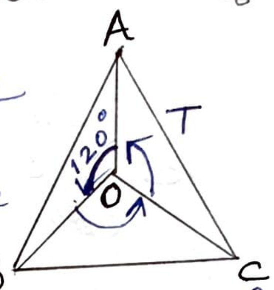
- A symmetry is a transformation that preserves both distances and the object under study.
- Properties of symmetry :—

(i) Composition of symmetries is a symmetry (closure).
(ii) " " " " associative.
(iii) Not doing anything is a symmetry (identity).
(iv) We can undo symmetries (inverse).

Therefore, symmetries of an object form a group.

Dihedral Group D_3. / Symmetries of an equilateral Triangle

Six symmetries of the equilateral triangle:



Let T be an equilateral triangle with its centroid O on a plane π and ρ be a rotation in the plane about O through $120°$.

Here ρ effects a permutation of the vertices but maps the triangle as a whole onto itself. ρ is a symmetry of the triangle T.

i : rotation in the plane about O through $0°$.

r_1 : " " " " O " $120°$.

r_2 : " " " " O " $240°$.

a : reflection about AO;

b : " BO;

c : " CO.

Let $S = \{i, r_1, r_2, a, b, c\}$. The composition (o) table is given as follows:

$i = \begin{pmatrix} A & B & C \\ A & B & C \end{pmatrix}$, $r_1 = \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix}$

$r_2 = \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix}$, $a = \begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix}$

$b = \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}$, $c = \begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix}$

o	i	r_1	r_2	a	b	c
i	i	r_1	r_2	a	b	c
r_1	r_1	r_2	i	c	a	b
r_2	r_2	i	r_1	b	c	a
a	a	b	c	i	r_1	r_2
b	b	c	a	r_2	i	r_1
c	c	a	b	r_1	r_2	i

It forms a non-abelian group.

inverse of $i = i$
" " $r_1 = r_2$
" " $r_2 = r_1$
" " $a = a$
" " $b = b$
" " $c = c$

This group is called the Dihedral group D_3, which is same as the symmetric group S_3.

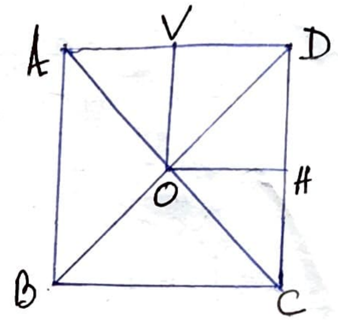
Note: Let S be the set of all points in a Euclidean space.

- An isometry of the space is a bijection of S onto S that preserves distance between two points in S.
- A symmetry of a geometrical figure in a E.S. is an isometry that keeps the figure as a whole unchanged.

Symmetries of a Square:

Dihedral group D_4 / Octic group.

Let us consider a square ABCD with centre at O. There are 8 symmetries of the square.



(i) Four rotations in the plane about O.

i : rotation through 0°

r_1 : " " 90°

r_2 : " " 180°

r_3 : " " 270°.

(ii) Four rotations out of the plane.

h : rotation about the horizontal line OH

v : " " " vertical " OV.

d : " " " principal diagonal OA.

d' : " " " other " OB.

Let $S = \{i, r_1, r_2, r_3, h, v, d, d'\}$

Composition table is given as follows:

$\circ$	i	r_1	r_2	r_3	h	v	d	d'
i	i	r_1	r_2	r_3	h	v	d	d'
r_1	r_1	r_2	r_3	i	d'	d	h	v
r_2	r_2	r_3	i	r_1	v	h	d'	d
r_3	r_3	i	r_1	r_2	d	d'	v	h
h	h	d	v	d'	i	r_2	r_1	r_3
v	v	d'	h	d	r_2	i	r_3	r_1
d	d	v	d'	h	r_3	r_1	i	r_2
d'	d'	h	d	v	r_1	r_3	r_2	i

$i = \begin{pmatrix} A & B & C & D \\ A & B & C & D \end{pmatrix}$, $r_1 = \begin{pmatrix} A & B & C & D \\ B & C & D & A \end{pmatrix}$

$r_2 = \begin{pmatrix} A & B & C & D \\ C & D & A & B \end{pmatrix}$, $r_3 = \begin{pmatrix} A & B & C & D \\ D & A & B & C \end{pmatrix}$

$h = \begin{pmatrix} A & B & C & D \\ B & A & D & C \end{pmatrix}$, $v = \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}$

$d = \begin{pmatrix} A & B & C & D \\ A & D & C & B \end{pmatrix}$, $d' = \begin{pmatrix} A & B & C & D \\ C & B & A & D \end{pmatrix}$

This is a non-commutative group and is called Dihedral group D_4.

Note: ① The symmetries of a regular pentagon form a non-commutative group of order 10, called the Dihedral group D_5.

② The symmetries of a regular n-gon form a non-abelian group of order $2n$, called Dihedral group D_n.

Properties of Symmetric group S_n of degree n.

① S_n is non-commutative group for $n \geq 3$.

S_3 " " " "

② $S_3 = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$. $o(S_3) = 3! = 6$.

$\rho_0 = (1)$, $\rho_1 = (1,2,3)$, $\rho_2 = (1,3,2)$, $\rho_3 = (2,3)$, $\rho_4 = (1,3)$, $\rho_5 = (1,2)$

$o(\rho_0) = 1$, $o(\rho_1) = 3$, $o(\rho_2) = 3$, $o(\rho_3) = o(\rho_4) = o(\rho_5) = 2$,

Since $\rho_0^1 = \rho_0$; $\rho_1^3 = \rho_1^2 \cdot \rho_1 = \rho_2 \cdot \rho_1 = \rho_0$; $\rho_2^3 = \rho_2^2 \cdot \rho_2 = \rho_1 \cdot \rho_2 = \rho_0$;

$\rho_3^2 = \rho_0 = \rho_4^2 = \rho_5^2$.

③ Subgroups of the group S_3:

$\{\rho_0, \rho_1, \rho_2\}$, $\{\rho_0, \rho_3\}$, $\{\rho_0, \rho_4\}$, $\{\rho_0, \rho_5\}$; since closure property is satisfied in each finite subsets.

④ Cyclic subgroups of S_3:

$\langle \rho_0 \rangle = \{\rho_0\}$; $\langle \rho_1 \rangle = \{\rho_0, \rho_1, \rho_2\}$, since $\rho_1^2 = \rho_2$, $\rho_1^3 = \rho_0$;

$\langle \rho_2 \rangle = \{\rho_0, \rho_1, \rho_2\}$, since $\rho_2^2 = \rho_1$, $\rho_2^3 = \rho_0$;

$\langle \rho_3 \rangle = \{\rho_0, \rho_3\}$, since $\rho_3^2 = \rho_0$;

$\langle \rho_4 \rangle = \{\rho_0, \rho_4\}$, since $\rho_4^2 = \rho_0$;

$\langle \rho_5 \rangle = \{\rho_0, \rho_5\}$, since $\rho_5^2 = \rho_0$.

All are proper subgroups of S_3.

⑤ S_3 is not cyclic, since $o(S_3) = 6$ and there exists no element of order 6 in S_3.

⑥ S_3 is not a cyclic group, since it is not abelian.

⑦ An abelian group is not necessarily a cyclic group. For example, Klein's 4-group V is abelian but not cyclic, as there exists no element of order 4 in V.

NOTE: Properties of the Dihedral group D_3 are same as that of the symmetric group S_3, as they can be treated as same group.

Properties of Dihedral group D_4 :-

① $D_4 = \{i, r_1, r_2, r_3, h, v, d, d'\}$. $O(D_4) = 8$.

② D_4 is a non-commutative group.

③ $O(r_1) = O(r_3) = 4$, $O(r_2) = O(h) = O(v) = O(d) = O(d') = 2$.

④ D_4 is not a cyclic group, since it is not abelian, also since there exists no element whose order is $8 = O(D_4)$.

⑤ Each of the finite subsets of D_4 : $\{i, r_1, r_2, r_3\}$, $\{i, r_2, h, v\}$, $\{i, r_2, d, d'\}$, $\{i, r_2\}$, $\{i, h\}$, $\{i, v\}$, $\{i, d\}$, $\{i, d'\}$ is closed w.r.t. 'multiplication of permutations'. Therefore, each is a proper subgroup of D_4.

Some more properties of the symmetric group S_n and the alternating group A_n :-

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⑧ A_n is a non-commutative group for $n \geq 4$.

⑨ A_3 is a commutative group.

⑩ A_3 is a commutative subgroup of the non-commutative group S_3.

⑪ A_3 is the only subgroup of order 3 of the group S_3.

⑫ If G be a commutative group then HK is a subgroup of G. But commutativity of G does not necessarily imply $HK = KH$. For example,
let $G = S_3$ (non-commutative), $H = \{e_0, (123), (132)\}$, $K = \{e_0, (23)\}$. Then $HK = \{e_0, (23), (123), (12), (132), (13)\}$, $KH = \{e_0, (123), (132), (23), (13), (12)\}$.
So $HK (= KH)$ is a subgroup of G.